2. BABESHKO V.A. and PRYAKHINA O.D., The fictitious absorption method in plane dynamic problems. Prikl. Mat. Mekh. 44, 3, 1980.
3. ALEKSANDROV V.M. and KOVALENKO E.V., Problems of the Mechanis of Continuous Media with Mixed Boundary Conditions, Nauka, Moscow, 1986.
4. ALEKSANDROV V.M. and CHEBAKOV M.I., On the method of homogeneous solutions in mixed problems of the theory of elasticity for a truncated wedge and an annular sector. Prikl. Mat. Mekh. 47, 5, 1983.
5. GRINCHENKO V.T. and MELESHKO V.V., Harmonic Oscillations and Waves in Elastic Bodies, Nauk. Dumka, Kiev, 1981.
6. GLUSHKOVA N.V., KALINCHUK V.V., SELEZNEV M.G. and others. Procedure and results of calculations of the wave field near a vibrational source located on the surface of a homogeneous semispace, In: Problems of the Excitation of Seismic waves by a Source of Vibrations Inst. Geol. i Geofiz. SO Akad. Nauk SSSR, Novosibirsk, 1976.

Translated by H.T.

PMM U.S.S.R.,Vol.52,No.5,pp.664-669,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ASYMPTOTIC PROPERTIES OF THE APPROXIMATE SOLUTION OF A CLASS OF DUAL INTEGRAL EQUATIONS*

S.M. AIZIKOVICH and I.S. TRUBCHIK

An investigation is presented of dual integral equations generated by various plane contact problems: a strip resting without friction on a rigid base (Problem 1), a strip clamped along the base (Problem 2), a wedge with a clamped face (Problem 3), and also be axisymmetric problems relating to the action of a ring-shaped stamp on a half-space (Problem 4), and the interaction of an elastic bandage with an elastic cylinder /1/ (Problem 5). The strip, wedge, half-space and cylinder may be uniform, laminar of continuously inhomogeneous. Analogous equations in terms of Laplace transforms are obtained in problems of coupled thermo-elasticity and consolidation theory of water-saturated media for the bodies listed here /2/.

The method described in /3/ is generalized to construct solutions of the above problems. Well-posedness and solvability classes are established for the equations, proving that the approximate method proposed here is asymptotic in both directions with respect to a characteristic geometric parameter $\lambda = H/a$ (H is the thickness of the strip and a is half the thickness of the stamp) in Problems 1 and 2, or $\lambda = 2/\ln(b/a)$ (a and b are the distances from the nearest and farthest points at which the stamp touches the boundary of the wedge to its vertex) in Problem 3, $\lambda = 2/\ln(b/a)$ (a is the inner radius of the stamp and b its outer radius) in Problem 4, $\lambda = R/a$ (R is the radius of the cylinder and a half the thickness of the bandage) in Problem 5. In problems of coupled thermo-elasticity and consolidation theory λ also involves the parameter p of the Laplace transform with respect to the time coordinate /2/. The method is illustrated in relation to a contact problem for a strip continuously inhomogeneous in depth.

1. Statement of the problem. Consider the dual integral equation

$$\int_{-\infty}^{\infty} \Phi(\alpha) \frac{\operatorname{th}(A\lambda\alpha)}{\alpha} L(\lambda\alpha) e^{-i\alpha x} d\alpha = 2\pi g(x), \quad |x| \leqslant 1 \tag{1.1}$$

---

*Prikl.Matem.Mekhan.,52,5, 850-856,1988

$$\int_{-\infty}^{\infty} \Phi(\alpha) e^{-i\alpha x} d\alpha = 0, \quad |x| > 1$$

Let $L(\lambda\alpha)$ possess the following properties:

$$L(\alpha) = B + C|\alpha| + o(\alpha^2), \quad \alpha \to 0 \tag{1.2}$$

$$L(\alpha) = 1 + D|\alpha|^{-1} + o(\alpha^{-2}), \quad \alpha \to \infty \tag{1.3}$$

where A, B, C and D are constants. Then by Theorem (1.1) of /4/ $L(\lambda\alpha)$ can be approximated by expressions of the form

$$L(\lambda\alpha) = L_N(\lambda\alpha) + L_M^{\Sigma}(\lambda\alpha) \tag{1.4}$$

$$L_N(\lambda\alpha) = \prod_{i=1}^{N} \frac{\alpha^2 + \delta_i^2\lambda^{-2}}{\alpha^2 + \gamma_i^2\lambda^{-2}}, \quad L_M^{\Sigma}(\lambda\alpha) = |\alpha| \sum_{k=1}^{M} \frac{d_k\lambda^{-1}}{\alpha^2 + \eta_k^2\lambda^{-2}}$$

Here δ_i, γ_i $(i = 1, \ldots, N)$, d_k, η_k $(k = 1, \ldots, M)$ are constants.

Definition 1. The function $L(\alpha)$ is in class Π_N (class Σ_M) if $L(\alpha) = L_N(\alpha)$ $(L(\alpha) = L_M^{\Sigma}(\alpha))$.

Definition 2. The function $L(\alpha)$ is in class $S_{N,M}$ if it can be expressed as

$$L(\alpha) = L_N(\alpha) + L_M^{\Sigma}(\alpha) \tag{1.5}$$

Let B_k denote the space of functions all of whose derivatives exist in $[-1, 1]$ up to order k inclusive, and moreover the k-th derivatives satisfy a Hölder condition with exponent $1/2 + \varepsilon$, $\varepsilon > 0$; endow this space with the usual norm /4/. Let C_γ^k denote the space of functions whose k-th derivatives are continuous with weight $(x+1)^\gamma(1-x)^\gamma$. The subspaces of C_γ^k of odd and even functions are denoted by C_γ^{k+} and C_γ^{k-}, respectively.

Eq.(1.1) is considered with the additional condition

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\alpha) e^{-i\alpha x} dx, \quad \int_{-1}^{1} \varphi(\xi) e^{-i\alpha\xi} d\xi = \Phi(\alpha) \tag{1.6}$$

The general problem (Problem S) of determining the function $\varphi(x)$ from the dual integral Eq.(1.1) with condition (1.6) reduces to solving two auxiliary problems: the "even" problem (Problem S^+) and the "odd" problem (Problem S^-), corresponding to resolution of the functions $g(x)$, $\varphi(x)$, $\Phi(\alpha)$ into even and odd terms (denoted below by plus and minus indices, respectively).

2. The existence and uniqueness of the solution of Eq.(1.1) for $L(\alpha)$ of class Π_N.

To solve Eq.(1.1) with $L(\lambda\alpha)$ of class Π_N and "even" $g_+(x)$, we use the lemmas from /5/.

Lemma 1. Assume that $g_+(x)$ has a Fourier series $\Sigma a_k \cos k\pi x$ in the interval $[-1, 1]$. Then the series $\Sigma k|a_k|$ converges if $g_+(x) \in B_1$.

Lemma 2. Under the assumptions of Lemma 1, the series $\Sigma k^2|a_k|$ converges if $g_+(x) \in B_2$.

The proof is analogous to the proof of Lemma 1 in /5/. The summation in Lemmas 1 and 2 is over k from 1 to ∞.

Lemma 3. Eq.(1.1) is uniquely solvable for $L(\alpha)$ of class Π_N if $g(x) = g_+(x) \in B_2$ in the class of functions $C_{1/2}^{0+}$. We then have an estimate

$$\|\varphi_+(x)\|_{C_{1/2}^{0+}} \leqslant m(\Pi_N) \|g_+\|_{B_2} \tag{2.1}$$

where $m(A)$ is a constant dependent on the specific form of the functions in class A.

Proof. Express the right-hand side of Eq.(1.1) as a Fourier series

$$g_+(x) = {}^1/_2 a_0 + a_1 \cos\pi x + a_2 \cos 2\pi x + \ldots \tag{2.2}$$

(This is always possible under the assumptions of Lemma 1). As in /6/, we obtain an expression for the stresses:

$$\varphi_+(x) = \frac{a_0}{2} \frac{\theta}{Q_{-1/2}(\operatorname{ch}\theta) K(1, x)} + \tag{2.3}$$

$$\sum_{k=1}^{\infty} a_k \frac{\operatorname{th}(A\lambda k\pi)}{L(\lambda k\pi)} F\left(-\frac{1}{2}, -\frac{1}{2} + i\frac{k\pi}{\theta}, x\right) +$$

$$\sum_{n=1}^{N} C_n F\left(-\frac{1}{2}, -\frac{1}{2}+\frac{\delta_n}{\lambda\theta}, x\right)$$

$$\theta = \pi/(A\lambda), \quad K(a, x) = \sqrt{2(\operatorname{ch}\theta a - \operatorname{ch}\theta x)}$$

$$F(u, v, x) = \theta \operatorname{sh}\theta \frac{P_v{}^1 Q_u - Q_u{}^1 P_v}{Q_u K(1, x)} -$$

$$\theta^2\left(v+\frac{1}{2}\right)^2 \int_x^1 \frac{P_v(\operatorname{ch}\theta\tau)}{K(\tau, x)} \operatorname{sh}\theta\tau\, d\tau$$

Here and below $P_u{}^\mu \equiv P_u{}^\mu(\operatorname{ch}\theta)$, $Q_u{}^\mu \equiv Q_u{}^\mu(\operatorname{ch}\theta)$ denote the associated Legendre functions of the first and second kinds, respectively.

The constants C_h are determined from the system of linear algebraic equations

$$\sum_{n=1}^{N} x_n = f_m + \sum_{n=1}^{N} a_{mn} x_n, \quad m = 1, \ldots, N \tag{2.4}$$

Here

$$x_n = Q_{-1/2} P_{-1/2+\delta_n/(\lambda\theta)} C_n, \quad a_{mn} =$$

$$\frac{Q_{-1/2}}{Q^1_{-1/2}} R\left(-\frac{1}{2}+\frac{\delta_n}{\lambda\theta}, -\frac{1}{2}+\frac{\gamma_m}{\lambda\theta}\right)$$

$$f_m = \frac{a_0}{2\operatorname{sh}\theta} + \sum_{k=1}^{\infty} a_k \frac{\operatorname{th}(A\lambda k\pi)}{L(\lambda k\pi)} \frac{(k\pi)^2}{Q_{-1/2+\gamma_m/(\lambda\theta)}} \times$$

$$E\left(-\frac{1}{2}+i\frac{k\pi}{\theta}, -\frac{1}{2}+\frac{\gamma_m}{\lambda\theta}, -\frac{1}{2}\right)$$

$$R(u, v) = \frac{(u+1/2)^2 P_u Q_v{}^1 - (v+1/2)^2 P_u{}^1 Q_v}{(u-v)(u+v+1) P_u Q_v}$$

$$E(u, v, w) = Q_w T(u, v) - Q_v T(u, w)$$

$$T(u, v) = \frac{P_u Q_v{}^1 - Q_v P_u{}^1}{((u+1/2)^2 - (v+1/2)^2)\theta^2}$$

We will estimate the expression on the right-hand side of (2.3). Expressions (2.3) and (2.4) are meaningful provided the series ocurring in them converge. Using the asymptotic properties of the Legendre functions /7/ and asymptotic estimates for the incomplete spherical functions in Poisson form (for a general definition of the incomplete hypergeometric functions see /8/), we see that the series in (2.4), (2.3) are bounded if the following series (summation over k from 1 to ∞) converge:

$$\Sigma(-1)^k k^{1/2} a_k, \ \Sigma(-1)^k k^{-1/2} a_k, \ \Sigma k^{3/2} a_k \tag{2.5}$$

The convergence of these series follows from Lemmas 1, 2 and the Leibnitz test for alternating series /9/. Hence we obtain (2.1).

A more general assertion can be established:

Lemma 4. Eq.(1.1) is uniquely solvable for $L(\lambda\alpha)$ of class Π_N if $g(x) = g_+(x) \in B_{k+2}$ in the class of functions $C^{k+}_{k+1/2}$. We then have an estimate

$$\| \varphi_+(x) \|_{C^{k+}_{k+1/2}} \leqslant m(\Pi_N, k) \| g_+ \|_{B_{k+2}} \tag{2.6}$$

The proof is analogous to that of Lemma 3, using estimates for the right-hand sides of (2.3) and (2.4).

The integral operator corresponding to a function $L(\alpha)$ of class X will also be denoted by X. Thus Eq.(1.1) for $L(\alpha) \in \Pi_N$ takes the form

$$\Pi_N \varphi = g_+ \tag{2.7}$$

As a corollary to Lemma 4, we obtain the following theorem.

Theorem 1. Under the assumptions of Lemma 4,

$$\| \varphi_+(x) \|_{C^{k+}_{k+1/2}} \leqslant \| \Pi_N^{-1} \| m(k) \| g_+ \|_{B_{k+2}} \tag{2.8}$$

Consider Problem S^-. Let us assume that the right-hand side of Eq.(1.1) can be expressed as a Fourier series

$$g_-(x) = \sum_{k=1}^{\infty} b_k \sin k\pi x \tag{2.9}$$

The Fourier coefficients in (2.9) can be so chosen that

$$g_+(x)=\int_0^x g_-(t)\,dt + C \tag{2.10}$$

where C satisfies the condition

$$\varphi_+(1) = 0 \tag{2.11}$$

In addition,

$$g_-(x) \in B_2, \quad \varepsilon < 1 \tag{2.12}$$

In that case, using the results of /10/, we can show that

$$\varphi_-(x) = \varphi_+'(x) \tag{2.13}$$

is a solution of the first integral Eq.(1.1), corresponding to $g_-(x)$. It follows from (2.13) and (2.3) that

$$\varphi_-(x)=-\sum_{k=1}^{\infty} b_k \frac{\operatorname{th}(A\lambda k\pi)}{L(\lambda k\pi)} k\pi\, S\left(-\frac{1}{2}+i\frac{k\pi}{\theta}, x\right) - \sum_{n=1}^{N} D_n\delta_n\lambda^{-1} S\left(-\frac{1}{2}+\frac{\delta n}{\lambda\theta}, x\right) \tag{2.14}$$

$$S(u,x)=\operatorname{sh}\theta x\left[\frac{P_u(\operatorname{ch}\theta)}{K(1,x)} - \theta\int_x^1 \frac{P_u^1(\operatorname{ch}\theta\tau)}{K(\tau,x)}\,d\tau\right]$$

The coefficients D_n are determined from the system of linear algebraic equations

$$\sum_{n=1}^{N} a_{mn}x_n = f_m, \quad m=1,2,\ldots,N \tag{2.15}$$

Here

$$x_n = D_n\delta_n\lambda^{-1}, \quad a_{mn}=T\left(-\frac{1}{2}+\frac{\delta_n}{\lambda\theta}, -\frac{1}{2}+\frac{\gamma_m}{\lambda\theta}\right)$$

$$f_m=\sum_{k=1}^{\infty} b_k \frac{\operatorname{th}(A\lambda k\pi)}{L(\lambda k\pi)} k\pi T\left(-\frac{1}{2}+i\frac{k\pi}{\theta}, -\frac{1}{2}+\frac{\gamma_m}{\lambda\theta}\right)$$

where $T(u,v)$ is defined in (2.4). Obviously, the solution of Eq.(1.1) for odd right-hand side belongs to class $C_{k+1/2}^{k-}$ if conditions (2.10)-(2.12) are satisfied, and in that case the estimate (2.8) holds with the plus sign replaced by minus.

This proposition is proved in the same way as Lemmas 4 and 3, using estimates for the right-hand sides of (2.14), (2.15).

By the superposition principle, the function $\varphi(x) = \varphi_+(x) + \varphi_-(x)$ is a solution of the integral Eq.(1.1) for the right-hand side $g(x)$ in the general case; hence, for $g(x) = g_+(x) + g_-(x)$ we have the following

Theorem 2. Eq.(1.1) is uniquely solvable for $L(\alpha)$ of class Π_N if $g(x) \in B_{k+2}$ in the class of functions $C_{k+1/2}^{k}$. We then have estimate (2.8), with the plus sign omitted.

3. Existence and uniqueness of the solution of Eq.(1.1) for $L(\alpha)$ of class $S_{N,M}$.

Eq.(1.1) may be written in operator form for $L(\lambda\alpha) \in S_{N,M}$:

$$\Pi_N\varphi + \Sigma_M\varphi = g \tag{3.1}$$

Lemma 5. The operator $\Pi_N^{-1}\Sigma_M$ of Problem S is a contracting operator in the space $C_{k+1/2}^{k}$ for $g(x) \in B_{k+2}$, if $0<\lambda<\lambda^*$ or $\lambda>\lambda^\circ$, where λ^*, λ° are certain fixed values of λ.

Proof. We prove the lemma for $k=0$. The proof for $k>0$ is similar. Consider the operator $\Sigma_M\varphi$. Without loss of generality, we can put $M=1$; we have

$$\Sigma_1\varphi = B\lambda^{-1}\pi\left[2\theta\sum_{l=1}^{\infty}(\eta^2-t_l^2)^{-1}\int_{-1}^{1}\varphi(\xi)\,e^{-t_l|\xi-x|}\,d\xi + \eta^{-1}\operatorname{tg}\frac{\pi\eta}{\theta}\int_{-1}^{1}\varphi(\xi)\,e^{-\eta|\xi-x|}\,d\xi\right] \tag{3.2}$$

$$\eta=\eta'\lambda^{-1}, \quad \theta=\pi/(A\lambda), \quad t_l={}^1/_2\theta(2l-1), \quad l=1,2,\ldots$$

(B, η' are constants).

Express $\Sigma_1\varphi$ as a series

$$\Sigma_1\varphi = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k^+ \cos k\pi x + c_k^- \sin k\pi x) \tag{3.3}$$

The coefficients $c_k^\pm$ are found from the formulae

$$c_k^\pm = 4\pi B\lambda^{-1}\left\{\left(2\theta\sum_{l=1}^{\infty}\frac{\tau}{t_l^2+k^2\pi^2}+\frac{t}{\eta^2+k^2\pi^2}\right)c_k^\pm + \right. \tag{3.4}$$

$$\left.(-1)^{k+1}\left[2\theta\sum_{l=1}^{\infty}\tau e(t_l)\int_0^1\varphi_\pm(\xi)\,h^\pm(t_l\xi)\,d\xi + te(\eta)\int_0^1\varphi_\pm(\xi)\,h^\pm(\eta\xi)\,d\xi\right]\right\}$$

$$\tau = \frac{t_l}{\eta^2+t_l^2}, \quad t = \operatorname{tg}\frac{\pi\eta}{\theta}, \quad e(a) = (a^2+\eta^2)\exp, a$$

$$h^+(a\xi) = \operatorname{ch} a\xi, \quad h^-(a\xi) = k\pi a^{-1}\operatorname{sh} a\xi$$

and using (3.4) we obtain the estimate

$$\max_{x\in[-1,1]}\left|\Sigma_1\varphi\sqrt{1-x^2}\right| \leqslant c_0 + \sum_{k=1}^{\infty}(|c_k^+|+|c_k^-|) \leqslant \lambda M^*, \ \lambda\to 0 \quad (\lambda<\lambda^*) \tag{3.5}$$

and the analogous estimate for $\lambda\to\infty$ $(\lambda>\lambda^\circ)$ when the right-hand side is replaced by $\lambda^{-1}M^\circ$, where M^*, M° are independent of λ. Therefore, using estimates analogous to those of Lemmas 3 and 4, we can choose λ in such a way that $\Pi_N^{-1}\Sigma_M$ is a contracting operator /11/ under the assumptions of the lemma.

By Lemma 5, applying the Banach contracting mapping principle to the equation

$$\varphi + \Pi_N^{-1}\Sigma_M\varphi = \Pi_N^{-1}g \tag{3.6}$$

we obtain a proof of the existence and uniqueness of the solution to Eq.(3.1) under the above assumptions.

Hence, from Theorem 2 we have the following:

Theorem 3. Eq.(1.1), with addditional conditions (1.2), (1.3) and (1.6), is uniquely solvable in the space $C_{k+1/2}^k$ if $g(x)\in B_{k+2}$ for $0<\lambda<\lambda^*$, $\lambda>\lambda^\circ$, where λ^*, λ° are certain fixed values of λ, and moreover

$$\|\varphi(x)\|_{C_{k+1/2}^k} \leqslant (\Pi_N, \Sigma_\infty, k)\,\|g(x)\|_{B_{k+2}} \tag{3.7}$$

The proof uses a well-known device of perturbation theory, based on successive approximations, just as in /12/.

4. *Example.* Let us consider Problem 1 for a homogeneous and a continuously inhomogeneous strip, whose Young's modulus varies with depth according to the law

$$E(z) = E_0 f(z), \quad z\in[-1, 0]$$
$$f(z) = 1{,}1 + \sin(k\pi z/2), \quad k = 1, 2, 3 \tag{4.1}$$

assuming a constant Poisson's ratio $\nu = 1/3$.

Fig.1 shows graphs of the transform of the kernel $L(\alpha)$ for a homogeneous layer (curve 0) and inhomogeneous layers with $k = 1, 2, 3$ (curves *1, 2, 3* respectively). The dashed curves 0, 1, 2, 3 correspond to the difference between the exact transform and its approximations according to (1.4).

Fig.2 presents graphs of the ratio $X(x) = \varphi_k(x)/\varphi_0(x)$, which characterizes the distribution of normal contact stresses $\varphi_k(x)$ under a flat-based stamp, pressed down with unit force on an inhomogeneous strip having Young's modulus (4.1), for different values of λ and $k=1$ (the solid curves), $k=2$ (the dashed curves) and $k=3$ (the dash-dotted curves). The quantity $\varphi_0(x)$ is the distribution of contact stresses under the stamp for an inhomogeneous strip with $E(z)=E_0$. The values of $\varphi_k(x)$ were found by formula (2.3) with $N=9$ $(k=1,2,3)$. The figures on the curves indicate λ values.

The results clearly show the increase in the singularity coefficient at the edge of the stamp, assuming that the strip has a monotonically decreasing modulus of elasticity. In the case of non-monotonic laws $(k=2, k=3)$, the curves $X(x)$ differently for small and large λ, indicating that the contact-pressure distribution depends significantly both on the thickness of the strip and on the nature of its inhomogeneity.

The authors are indebted to V.M. Aleksandrov for his interest.

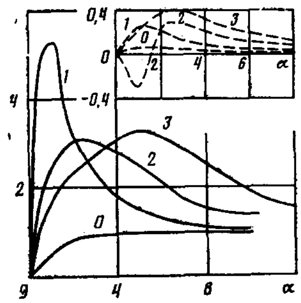

Fig.1

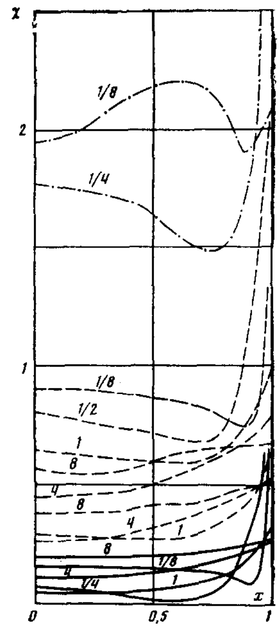

Fig.2

REFERENCES

1. ALEKSANDROV V.M. and ROMALIS B.L., Contact Problems in Mechanical Engineering, Mashinostroyenie, Moscow, 1986.
2. KERCHMAN V.I., Problems of consolidation and coupled thermoelasticity for a deformed half-space. Izv. Akad. Nauk SSSR, MTT, 1, 1976.
3. ALEKSANDROV V.M., On the solution of a class of dual equations. Dokl. Akad. Nauk SSSR, 210, 1, 1973.
4. AIZIKOVICH S.M., Asymptotic solution of contact problems of elasticity theory for media inhomogeneous in depth. Prikl. Mat. Mekh., 46, 1, 1982.
5. ZYGMUND A., Trigonometric series, I-II. Cambridge University Press, Cambridge, 1959.
6. ZELENTSOVA V.B., On the solution of a certain class of integral equations, Prikl. Mat.Mekh., 46, 5, 1982.
7. HOBSON E.W., The Theory of Spherical and Ellipsoidal Harmonics. Cambridge University Press, Cambridge, 1931.
8. AGREST M.I. and MAKSIMOV M.Z., Theory of Incomplete Cylindrical Functions and Their Applications, Atomizdat, Moscow, 1965.
9. FIKHTENGOL'TS G.M., A Course of Differential and Integral Calculus, Nauka, Moscow, 1969.
10. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of Elasticity Theory, Nauka, Moscow, 1974.
11. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis, Nauka, Moscow, 1977.
12. BABESHKO V.A., Convolution-type integral equations of the first kind on a system of segments, arising in elasticity theory and mathematical physics. Prikl. Mat. Mekh., 35, 1, 1971.

Translated by D.L.